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# MULTICOVER INEQUALITIES ON COLORED COMPLEXES

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Suppose that $C$ is a balanced simplicial complex. We show that its flag $f$-vector contains an interesting multiplicative structure. We define $\eta_S(C) := \log_2 f_S(C)$, and characterize the convex cone in which this flag $\eta$-vector may lie. Additionally, we specialize our results to the case when $C$ is a pure balanced simplicial complex, and when $C$ is a graded poset.

## 1. Introduction

In an ordinary simplicial complex, one may count the number of faces of each dimension, i.e. how many one-dimensional faces there are, how many two-dimensional faces there are, and so on. In so doing, we associate a vector of numbers to each simplicial complex, called the $f$-vector of the complex. Now, not every vector of numbers is the $f$-vector of a simplicial complex. In the 1960s, Kruskal and Katona independently discovered a purely numeric characterization of which vectors are $f$-vectors of simplicial complexes [4,3].

A simplicial complex is said to be $k$-colorable if the graph formed by its vertexes and edges is $k$-colorable in the usual sense for graphs; a choice of a coloring yields a $k$-colored simplicial complex. We may again tally the number of faces of each dimension, and ask the same question; namely, which vectors of numbers are the $f$-vectors of a colored complex. This question was answered in 1988, when Frankl, Füredi, and Kalai gave a numerical characterization of the $f$-vector [2], whose content and techniques are quite similar to that of Katona.

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If, instead of tallying faces by their dimension, we count them by which colors they contain, then we obtain the flag $f$-vector of the colored complex. We investigate the question of which collections of numbers can be the flag $f$-vector of a balanced, or pure balanced simplicial complex. In particular, we focus on the linear structure of the flag $f$-vector and a related invariant, the flag $\eta$-vector.

While the flag $f$-vector of a balanced simplicial complex proves to have no interesting linear structure, it has a rich multiplicative structure. To explore this structure, we take logarithms, turning questions about multiplication into questions about addition.

As main results, we prove the following two theorems:

**Theorem 1.** *Let $H$ be the closure of the convex hull of the set*

$$\{(\log f_S(\Delta))_{S\subseteq[k]} | \Delta \text{ is a } (k-1)\text{-dimensional balanced simplicial complex}\}$$

*of all flag $\eta$-vectors. Then $v \in H$ if and only if the following conditions hold:*

1. $v_S \geq 0$ *for all* $S \subseteq [k]$,
2. $v_\emptyset = 0$,
3. $m_S(\mathfrak{M}) v_S \leq \sum_{M\in\mathfrak{M}} v_M$ *for all* $S \subseteq [k]$ *and all multicovers* $\mathfrak{M}$ *of* $S$.

*In particular, $H$ is a convex cone with vertex at the origin.*

**Theorem 2.** *Suppose $I \subseteq \mathbb{R}_+^{2^{[k]}}$. The following are equivalent:*

1. *$I$ is the closure of the convex hull of the set*

$$\{\log f(\Delta) : \Delta \text{ is a } (k-1)\text{-dimensional pure balanced simplicial complex}\}.$$

2. *$I$ is the closure of the convex hull of the set*

$$\{\log f(P) : P \text{ is a rank } k+1 \text{ graded poset}\}.$$

3. *$I$ is the set of points $v$ which satisfy:*
   (a) $v_S \geq 0$ *for all* $S \subseteq [k]$,
   (b) $v_\emptyset = 0$,
   (c) $v_T \leq v_S$ *whenever* $T \subseteq S$, *and*
   (d) $m(\mathfrak{M}) v_{\cup\mathfrak{M}} \leq \sum_{M\in\mathfrak{M}} v_M$ *for every multicollection* $\mathfrak{M}$ *of subsets of* $[k]$.

*In particular, in each case, $I$ is a cone with vertex at the origin.*

## 2. Definitions

Let $k$ be an arbitrary integer, and $S$ be an arbitrary set of integers. For the purposes of this article, we define the following common sets and functions:

$$\begin{aligned} \mathbb{N} &:= \{0, 1, 2, 3, \ldots\}; \\ \mathbb{Z}_+ &:= \{1, 2, 3, \ldots\}; \\ [k] &:= \{1, 2, \ldots, k\}; \\ \iota_S(k) &:= \begin{cases} 1 & \text{if } k \in S, \\ 0 & \text{if } k \notin S. \end{cases} \end{aligned}$$

A *coloring* of an abstract simplicial complex $\Delta$ is a map $c_\Delta$ which assigns to each vertex of $\Delta$ a positive integer (a "color") in such a way that no face of $\Delta$ contains two vertexes with the same color. A coloring $c_\Delta$ is said to be a $k$-coloring of $\Delta$ if $c_\Delta(v) \in [k]$ for all vertexes $v$ of $\Delta$. We say that a simplicial complex $\Delta$ is $k$-colorable if there exists a $k$-coloring of the complex, and $k$-colored if $\Delta$ comes equipped with a given $k$-coloring. It is easy to see that a $(k-1)$-dimensional complex must be colored with at least $k$ colors. A colored complex which meets this bound, i.e. a $k$-colored complex which is $(k-1)$-dimensional is said to be balanced.

A special type of balanced complex will be of particular interest to us. We define the *complete complex* $K(a_1, \ldots, a_k)$ to be the complex which contains $a_1$ vertexes of color 1, $a_2$ vertexes of color 2, etc., and which contains every possible simplex with these vertexes (i.e. so that no simplex contains two vertexes of the same color).

Given a $k$-colored complex $\Delta$, and a set $S \subseteq [k]$, define $f_S(\Delta) = |\{\sigma \in \Delta | c_\Delta(\sigma) = S\}|$. The function which given $S$ yields $f_S(\Delta)$ is known as the *flag $f$-vector* of $\Delta$.

## 3. The flag $f$-vector and linear inequalities

The flag $f$-vectors of balanced simplicial complexes have no interesting linear structure:

**Theorem 3.1.** *Suppose that for some set of coefficients $\{\alpha_S\} \in \mathbb{R}$ we have $\sum_{S \subseteq [k]} \alpha_S f_S(\Delta) \geq 0$ for all balanced $(k-1)$-dimensional complexes $\Delta$. Then $\alpha_S \geq 0$ for all $S \subseteq [k]$.*

**Proof.** The mechanics of this proof are easier to see if we first broaden our range to allow for arbitrary $k$-colored complexes, so we will first show that result.

Suppose to the contrary that there is a set of coefficients satisfying the hypotheses that contains a negative element. Choose $T \subseteq [k]$ of minimum size such that $\alpha_T < 0$. Let $\Delta_i = K(i \cdot \iota_T(1), i \cdot \iota_T(2), \dots, i \cdot \iota_T(k))$. Then $\Delta_i$ is a $|T|$-colored colored complex with $f_S(\Delta_i) = i^{|S|}$ if $S \subseteq T$ and $f_S(\Delta_i) = 0$ otherwise.

Thus $\sum_{S\subseteq[k]} \alpha_S f_S(\Delta_i)$ is a polynomial in $i$ with leading term $\alpha_T i^{|T|}$. As $\alpha_T < 0$, there exists $i$ sufficiently large so that $\sum_{S\subseteq[k]} \alpha_S f_S(\Delta_i) < 0$. So $\Delta_i$ is a witness against our hypothesis that $\sum_{S\subseteq[k]} \alpha_S f_S(\Delta) \geq 0$ for all $k$-colored complexes $\Delta$.

To show the same result for balanced complexes, replace $\Delta_i$ with $\Sigma_i = \Delta_i \cup K(1,1,\dots,1)$. Then we have $f_S(\Sigma_i) = 1 + f_S(\Delta_i)$. We make two observations about this new complex. First $\Sigma_i$ is a $(k-1)$ dimensional balanced complex for each $i \geq 1$. Second, $\sum_{S\subseteq[k]} \alpha_S f_S(\Sigma_i)$ remains a polynomial in $i$ with leading term $\alpha_T i^{|T|}$, and so the same contradiction arises. ∎

## 4. The flag $\eta$-vector and Multicover inequalities

On the other hand, the flag $f$-vector of every colored simplicial complex, and therefore every balanced simplicial complexes, has an elegant multiplicative structure. To describe that structure, we need to first define the concept of a multicover.

### 4.1. Multicover inequalities

If $S$ is a finite set, a *multicover* of $S$ is a multicollection of subsets of $S$ whose union is $S$; by multicollection, we mean a collection (set) where repeats are allowed. If $\mathfrak{M}$ is a multicover of $S$, and $i \in S$, we define $n_i(\mathfrak{M})$ to be the number of elements of $\mathfrak{M}$ which contain $i$; in making this count, any repeated candidate is counted multiple times.

Let $S \subseteq [k]$, and $\mathfrak{M}$ be a multicover of $S$. We define the *minimal covering number* of $\mathfrak{M}$ in $S$ to be $m_S(\mathfrak{M}) = \min_{i\in S} n_i(\mathfrak{M})$. If $\mathfrak{M}$ is some arbitrary multicollection of subsets of $[k]$, we define $\cup\mathfrak{M}$ to be the set $\bigcup_{M\in\mathfrak{M}} M$; then $\mathfrak{M}$ is a multicover of $\cup\mathfrak{M}$. We will often make the abbreviation $m(\mathfrak{M}) = m_{\cup\mathfrak{M}}(\mathfrak{M})$.

**Theorem 4.1.** *Suppose that $\Delta$ is a $k$-colored simplicial complex, $S \subseteq [k]$, and $\mathfrak{M}$ is a multicover of $S$. Then*

$$f_S^{m_S(\mathfrak{M})}(\Delta) \leq \prod_{M\in\mathfrak{M}} f_M(\Delta).$$

**Proof.** This is a trivial application of Shearer's Entropy Lemma [1]. Let $V$ be the vertex set of $\Delta$, and $E$ be the collection of $S$-colored simplexes in $\Delta$. Then we may form the hypergraph $H=(V,E)$. For each $M\in\mathfrak{M}$, let $A_M$ be the collection of vertexes of $V$ whose color is contained in $M$. Then, for each $M\in\mathfrak{M}$, the collection $E_M$ of $M$-colored simplexes of $\Delta$ is exactly the edge set of the induced hypergraph $H[A_M]$. By Shearer's Lemma,

$$|E|^{m_S(\mathfrak{M})} \leq \prod_{M\in\mathfrak{M}} |E_M|,$$

which is exactly the conclusion of the Theorem. ∎

## 5. Balanced complexes and the flag $\eta$-vector

### 5.1. The flag $\eta$-vector

The strict positivity of the flag $f$-vector of a balanced complex allows us to view the multicover inequalities on balanced complexes in a completely new light. By taking logarithms, the multiplicative inequalities of Theorem 4.1 can be transformed into linear inequalities. This transformation allows us to apply the wealth of tools available for linear problems to our advantage. To this end, let $\eta_S(\Delta):=\log_2 f_S(\Delta)$; we call the collection $(\eta_S(\Delta))_{S\subseteq[k]}$ the *flag $\eta$-vector* of $\Delta$. Applying this definition to Theorem 4.1 yields:

**Corollary 5.1.** *Suppose $\Delta$ is a $(k-1)$-dimensional balanced simplicial complex, and $S\subseteq[k]$. Then for each multicover $\mathfrak{M}$ of $S$ we have*

$$m_S(\mathfrak{M})\eta_S(\Delta) \leq \sum_{M\in\mathfrak{M}} \eta_M(\Delta).$$ ∎

We will call these inequalities the (linear) *multicover inequalities.*

### 5.2. The cone of flag $\eta$-vectors and polynomial growth

A natural question to ask is if the set described by Corollary 5.1 is the smallest convex set containing all flag $\eta$-vectors. This is essentially true; we need only add the inequalities which insist on the non-negativity of the flag $\eta$-vector.

**Theorem 5.2.** *Let $H$ be the closure of the convex hull of the set*

$$\{(\eta_S(\Delta))_{S\subseteq[k]} | \Delta \text{ is a } (k-1)\text{-dimensional balanced simplicial complex}\}$$

*of all flag $\eta$-vectors. Then $v\in H$ if and only if the following conditions hold:*

*1.* $v_S \geq 0$ *for all* $S \subseteq [k]$,
*2.* $v_\emptyset = 0$,
*3.* $m_S(\mathfrak{M}) v_S \leq \sum_{M \in \mathfrak{M}} v_M$ *for all* $S \subseteq [k]$ *and all multicovers* $\mathfrak{M}$ *of* $S$.

*In particular,* $H$ *is a convex cone, with its vertex at the origin.*

The "only if" implication is just a restatement of 5.1. To prove the "if" implication, we need to look a bit deeper into the structure of the set $H$.

**Lemma 5.3.** *Suppose* $v \in \mathbb{N}^{2^{[k]}}$ *satisfies* $v_\emptyset = 0$ *and* $m_S(\mathfrak{M}) v_S \leq \sum_{M \in \mathfrak{M}} v_M$ *for all* $S \subseteq [k]$ *and* $\mathfrak{M}$ *a multicover of* $S$*. Then there exists a sequence* $\Delta_2, \Delta_3, \ldots$ *of* $(k-1)$*-dimensional balanced simplicial complexes and a positive integer* $\omega$ *such that*

$$\lim_{i \to \infty} \frac{\eta_S(\Delta_i)}{\log_2 i} = \omega v_S$$

*for all* $S \subseteq [k]$.

**Proof.** Given $k$-colored complexes $\Delta$ and $\Gamma$, define their disjoint union $\Delta \sqcup \Gamma$ in the usual way. Note that $f_S(\Delta \sqcup \Gamma) = f_S(\Delta) + f_S(\Gamma)$ unless $S = \emptyset$.

*(Step 1)* Let $\omega$ be an arbitrary positive integer for the time being. We will determine it later. For each $S \subseteq [k]$, define

$$\mathfrak{X}_S^\omega := \left\{ x \in \mathbb{N}^k : \sum_{j \in T} x_j \leq \omega v_T \text{ for all } T \subseteq S \right\}.$$

Clearly the zero vector is in $\mathfrak{X}_S^\omega$ for each $S$, so these sets are non-empty. For each $i \geq 2$, define

$$\Delta_i^\omega := \bigsqcup_{S \subseteq [k], x \in \mathfrak{X}_S^\omega} K\big(\iota_S(1) i^{x_1}, \iota_S(2) i^{x_2}, \ldots, \iota_S(k) i^{x_k}\big).$$

We claim this sequence of complexes has the desired properties.

Trivially we have that $\eta_\emptyset(\Delta_i^\omega) = 0$ for each $i$.

Let $S \neq \emptyset$ be chosen arbitrarily and fixed. We have that

$$f_S\big(K\big(\iota_S(1) i^{x_1}, \ldots, \iota_S(k) i^{x_k}\big)\big) = \prod_{j \in S} i^{x_j} = i^{\sum_{j \in S} x_j}.$$

Thus $f_S(\Delta_i^\omega)$ is a polynomial in $i$. For each $R \subseteq [k]$ with $S \subseteq R$, and each $x \in \mathfrak{X}_R^\omega$, there is a term of degree $\sum_{j \in S} x_j$. The leading term of $f_S(\Delta_i^\omega)$ therefore has degree

$$\max_{\substack{x \in \mathfrak{X}_R^\omega \\ S \subseteq R \subseteq [k]}} \sum_{j \in S} x_j.$$

Now, if $R \supseteq S$ then $\mathfrak{X}^\omega_R \subseteq \mathfrak{X}^\omega_S$, so we may in fact choose $x \in \mathfrak{X}^\omega_S$. It follows that $f_S(\Delta^\omega_i)$ is a polynomial in $i$ of degree $\max_{x\in\mathfrak{X}^\omega_S} \sum_{j\in S} x_j$. So

$$\lim_{i\to\infty} \frac{\eta_S(\Delta^\omega_i)}{\log_2 i} = \max_{x\in\mathfrak{X}^\omega_S} \sum_{j\in S} x_j.$$

Now by construction $\sum_{j\in S} x_j \leq \omega v_S$ for every $x \in \mathfrak{X}^\omega_S$. Thus

$$\lim_{i\to\infty} \frac{\eta_S(\Delta^\omega_i)}{\log_2 i} \leq \omega v_S \text{ for every } \omega \in \mathbb{Z}_+.$$

We need only find an $\omega$ for which the reverse inequality holds.

*(Step 2)* We wish to solve the integer program whose objective is to maximize $\sum_{j\in S} x_j$ over $x \in \mathfrak{X}^\omega_S$. For the moment, let us consider a simpler problem.

Let $\mathbb{Q}_+$ denote the non-negative rationals, and define

$$\mathscr{Z}_S := \left\{ z \in \mathbb{Q}^k_+ : \sum_{j\in T} z_j \leq v_T \text{ for all } T \subseteq S \right\}.$$

We consider the program whose objective is to maximize $\sum_{j\in S} z_j$ over all $z \in \mathscr{Z}_S$. This is a linear program over the rationals that serves as a simplification of the previous integer program. Note that the omission of $\omega$ is intentional; we will define $\omega$ in such a way that will allow us to take a solution to our rational program and derive a solution to our integer program.

By duality of linear programs (see e.g. [5]), we may instead consider the linear program whose domain is

$$\mathscr{Y}_S := \left\{ y \in \mathbb{Q}^{2^{|S|}}_+ : \sum_{j\in T\subseteq S} y_T \geq 1 \text{ for each } j \in S \right\}$$

where the objective is to minimize $\sum_{T\subseteq S} y_T v_T$, so long as they both have feasible solutions. It is not difficult to check that $z = (0,\ldots,0)$ and $y = (1,\ldots,1)$ are feasible solutions. By this duality, the minimum of $\sum_{T\subseteq S} y_T v_T$ over $y \in \mathscr{Y}_S$ is the same as the maximum of $\sum_{j\in S} z_j$ over $z \in \mathscr{Z}_S$.

Suppose that $y \in \mathscr{Y}_S$. We give the following interpretation of $y$: Let $m$ be the least common denominator of the entries of $y$, so that $m\cdot y$ is an integral vector. Let $\mathfrak{M}$ be the multicollection containing each subset $T \subseteq S$ with multiplicity $m\cdot y_T$. Then the statement that $\sum_{j\in T\subseteq S} y_T \geq 1$, or equivalently $\sum_{j\in T\subseteq S} m \cdot y_T \geq m$, says that there are at least $m$ sets in $\mathfrak{M}$ (counting repetitions) which contain $j$. Thus $m_S(\mathfrak{M}) \geq m$.

So by the multicover inequalities, we have

$$\begin{aligned} mv_S &\leq m_S(\mathfrak{M})v_S \\ &\leq \sum_{M\in\mathfrak{M}} v_M \\ &= \sum_{T\subseteq S}(m\cdot y_T)v_T. \end{aligned}$$

Thus $\sum_{T\subseteq S} y_T v_T \geq v_S$. This holds for each $y\in\mathscr{Y}_S$m. In particular,

$$\min_{y\in\mathscr{Y}_S}\sum_{T\subseteq S} y_T v_T \geq v_S.$$

By duality, this means

$$\max_{z\in\mathscr{X}_S}\sum_{j\in S} z_j \geq v_S.$$

Choose a particular $z\in\mathscr{X}_S$ which maximizes $\sum_{j\in S} z_j$. Let $\ell_S$ be the least common denominator of the entries of $z$, so that $\ell_S z$ is an integral vector. Suppose that $\ell_S$ divides $\omega$. Then $\omega z$ is also integral. Since $z\in\mathscr{X}_S$, we have $\omega z\in\mathfrak{X}_S^\omega$. Additionally, we have $\sum_{j\in S}\omega z_j\geq\omega v_S$.

*(Step 3)* Thus we have that

$$\max_{x\in\mathfrak{X}_S^\omega}\sum_{j\in S} x_j \geq \omega v_S$$

whenever $\omega$ is divisible by $\ell_S$. Recalling step 1, we then have

$$\lim_{i\to\infty}\frac{\eta_S(\Delta_i^\omega)}{\log_2 i} = \omega v_S$$

for every $S$ for which $\ell_S$ divides $\omega$. We stated at the beginning that $\omega$ would be determined later; we give its value now. If we let $\omega := \prod_{S\subseteq[k]}\ell_S$, and $\Delta_i = \Delta_i^\omega$, we have the desired result. ∎

Before showing how this lemma yields Theorem 5.2 we stop to mention a note about this proof. While we do not make this result explicit, this proof shows that the cone $H$ is in fact polyhedral: it has a finite number of facets. Roughly speaking, facets of this cone correspond to extreme points of $\mathscr{Y}_S$; there are clearly only a finite number of them for each $S$, and only a finite number of subsets $S\subseteq[k]$.

The conclusion of the proof of Theorem 5.2 is a fairly standard analytical argument.

**Proof of Theorem 5.2.**

($\Rightarrow$) This is just a restatement of Corollary 5.1.

($\Leftarrow$) We first show that the projection $H$ omitting the $S=0$ coordinate lies on no hyperplane. Suppose to the contrary that $\sum_{\emptyset\subsetneq S\subseteq[k]}\alpha_S\eta_S(\Delta)=c$ for some $c\in\mathbb{R}$, some $(\alpha_S)_{S\subseteq[k]}$, and all $(k-1)$-dimensional balanced complexes $\Delta$. Let $\Gamma_i = K(2^i, 2^i, \ldots, 2^i)$. Then $\eta_S(\Gamma_i)=i^{|S|}$, except when $S=\emptyset$, where we have $\eta_\emptyset(\Gamma_i)=0$. Hence $\sum_{S\subseteq[k]}\alpha_S\eta_S(\Gamma_i)$ is a polynomial with 0 constant term. In particular, the only way this could be equal to $c$ for all $i$ is if $\alpha_S=0$ for all $S\neq\emptyset$. It is interesting to note that slightly more is proven here than stated: in fact, if we were to restrict our attention to only pure complexes or even order complexes of graded posets, the same statement would hold, as $\Gamma_i$ is the (pure) order complex of a graded poset for each $i$.

Let $\varepsilon > 0$ and $v$ satisfying the noted inequalities be given. Choose a rational point $p\in H$ so that

$$\|p-v\|_\infty < \frac{\varepsilon}{2}. \tag{1}$$

We may do this as $H$ lies on no hyperplane, and so is the closure of its (nonempty) interior. Let $n$ be the least common denominator of $p$, so that $np$ is integral. Then there exists by Lemma 5.3 a sequence $\Delta_2, \Delta_3, \ldots$ of balanced $k$-colored simplicial complexes and a $\omega\in\mathbb{N}$ such that

$$\lim_{i\to\infty}\frac{\eta_S(\Delta_i)}{\log_2 i} = \omega n p_S$$

for each $S\subseteq[k]$. Choose $i$ sufficiently large to ensure that

$$\left\|\frac{\eta(\Delta_i)}{\log_2 i} - \omega n p\right\|_\infty < \frac{\varepsilon}{2}.$$

Then we have that

$$\left\|\left(\frac{1}{\omega n\log_2 i}\right)\eta(\Delta_i) - p\right\|_\infty < \frac{\varepsilon}{2\omega n} \leq \frac{\varepsilon}{2}. \tag{2}$$

Let $\lambda := \frac{1}{\omega n\log_2 i}$. Then we have by the triangle inequality, (1), and (2) that

$$\left\|\lambda\eta(\Delta_i) - v\right\| < \varepsilon.$$

We make two notes here. First, $\lambda\in[0,1]$ by construction. Second, the $(k-1)$-simplex, properly colored, has as its flag $\eta$-vector the zero vector. Thus $\lambda\eta(\Delta_i)=\lambda\eta(\Delta_i)+(1-\lambda)0$ is the convex combination of the flag $\eta$-vectors of two $(k-1)$-dimensional balanced simplicial complexes. And so $v$ lies in the closure of the convex hull of the set of all flag $\eta$-vectors of $(k-1)$-dimensional balanced simplicial complexes. That is, $v\in H$. ∎

### 5.3. Multiplicative inequalities on balanced complexes

The characterization of the linear structure of flag $\eta$-vectors yields as an obvious corollary the following:

**Corollary 5.4.** *Any inequality of the form $\prod_{S\subseteq[k]} f_S(\Delta)^{\alpha_S} \leq c$ which holds for all $(k-1)$-dimensional balanced simplicial complexes $\Delta$ follows as a consequence of the multicover inequalities of Theorem 4.1 and the positivity of the flag $f$-vector.* ∎

It is worth noting that for any such inequality, we may always substitute $c=1$ without changing the validity of the inequality; this is essentially because $H$ is a cone, i.e., with its vertex at the origin.

## 6. Multicover inequalities on pure balanced complexes

An important subclass of the balanced complexes are the pure balanced complexes. Quite often pure complexes pose a larger challenge to combinatorial analysis than their more generic counterparts. This is not so in our case, though. The only additional inequalities that hold in the pure balanced case are those dictating the monotonicity of the flag $f$-vector; that is, if $S\supseteq T$ then $f_S(\Delta)\geq f_T(\Delta)$. In analogy to Theorem 5.2, we have:

**Theorem 6.1.** *Let $I$ be the closure of the convex hull of the set $\{\eta(\Delta):$ $\Delta$ is a $(k-1)$-dimensional pure balanced simplicial complex$\}$. Then $v\in I$ if and only if:*

1. *$v_S\geq 0$ for all $S\subseteq[k]$,*
2. *$v_\emptyset=0$,*
3. *(Monotonicity) $v_T\leq v_S$ whenever $T\subseteq S$, and*
4. *(Multicover) $m(\mathfrak{M})v_{\cup\mathfrak{M}}\leq\sum_{M\in\mathfrak{M}} v_M$ for every multicollection $\mathfrak{M}$ of subsets of $[k]$.*

**Proof.** ($\Rightarrow$) That conditions (1), (2) hold is obvious; condition (4) holds due to Corollary 5.1. We need only show condition (3).

Let $\Delta$ be a $(k-1)$-dimensional pure balance simplicial complex. Suppose $\sigma\in\Delta$ and $c(\sigma)=T$. If $T\subseteq S$, then there must exist some simplex $\tau\in\Delta$ with $c(\tau)=S$ and $\sigma\subseteq\tau$. This follows because $\Delta$ is pure. For each $\sigma\in\Delta$ with $c(\sigma)=T$, choose some such $\tau$. Since each simplex of color signature $S$ contains exactly one simplex of color signature $T$, this assignment is an injection. This shows that $f_T(\Delta)\leq f_S(\Delta)$. So $\eta_T(\Delta)\leq\eta_S(\Delta)$.

($\Leftarrow$) To show the reverse direction, we follow exactly the outline as in the non-pure case. The only thing that needs change is our argument in Lemma 5.3.

The fact that we wish to construct a pure complex allows us to give a slightly simpler construction than in Lemma 5.3; we need only specify the $(k-1)$-dimensional simplexes.

Let

$$\begin{aligned}\mathfrak{X}^{\omega}_{[k]} &:= \left\{ x \in \mathbb{N}^k : \sum_{j \in T} x_j \leq \omega v_T \text{ for all } T \subseteq [k] \right\},\\ \varDelta^{\omega}_i &:= \bigsqcup_{x \in \mathfrak{X}^{\omega}_{[k]}} K(i^{x_1}, i^{x_2}, \dots, i^{x_k}).\end{aligned}$$

Arguing as before, we have

$$\lim_{i \to \infty} \frac{\eta_S(\varDelta^{\omega}_i)}{\log_2 i} = \max_{x \in \mathfrak{X}^{\omega}_{[k]}} \sum_{j \in S} x_j,$$

and by construction

$$\max_{x \in \mathfrak{X}^{\omega}_{[k]}} \sum_{j \in S} x_j \leq \omega v_S \text{ for all } \omega \in \mathbb{Z}_+.$$

On the other hand, for a properly chosen $\omega$,

$$\max_{x \in \mathfrak{X}^{\omega}_{[k]}} \sum_{j \in S} x_j \geq \omega v_{[k]},$$

by step 2 of Lemma 5.3. By the monotonicity hypothesis, $v_{[k]} \geq v_S$ for each $S \subseteq [k]$. So

$$\max_{x \in \mathfrak{X}^{\omega}_{[k]}} \sum_{j \in S} x_j = \omega v_S \text{ for each } S \subseteq [k].$$

Thus for the proper choice of $\omega$ we have the sequence $\varDelta^{\omega}_2, \varDelta^{\omega}_3, \dots$ satisfying

$$\lim_{i \to \infty} \frac{\eta_S(\varDelta^{\omega}_i)}{\log_2 i} = \omega v_S \text{ for all } S \subseteq [k].$$

The remainder of the argument is identical to the earlier case. ∎

## 7. Multicover inequalities on graded posets

The proof of Theorem 6.1 shows somewhat more than is stated. The complexes $\Delta_i^\omega$ constructed therein are in fact order complexes of graded posets: any complete complex is the order complex of a graded poset, and the disjoint union of two order complexes of the same dimension is again an order complex.

If we define $\eta(P)$ for a given graded poset $P$ to be $\eta(\Delta)$ where $\Delta$ is the order complex of $P$, then we may sum this observation up as follows:

**Theorem 7.1.** *Suppose $I \subseteq \mathbb{R}_+^{2^{[k]}}$. The following are equivalent:*

1. *$I$ is the closure of the convex hull of the set*
$$\{\eta(\Delta) : \Delta \text{ is a } (k-1)\text{-dimensional pure balanced simplicial complex}\}.$$
2. *$I$ is the closure of the convex hull of the set*
$$\{\eta(P) : P \text{ is a rank } k+1 \text{ graded poset}\}.$$
3. *$I$ is the set of points $v$ which satisfy:*
   (a) *$v_S \geq 0$ for all $S \subseteq [k]$,*
   (b) *$v_\emptyset = 0$,*
   (c) *$v_T \leq v_S$ whenever $T \subseteq S$, and*
   (d) *$m(\mathfrak{M}) v_{\cup\mathfrak{M}} \leq \sum_{M\in\mathfrak{M}} v_M$ for every multicollection $\mathfrak{M}$ of subsets of $[k]$.* ∎

## 8. Directions for further study

**Problem 8.1.** In addition to spanning a cone, one can show that the set of flag $\eta$-vectors of $(k–1)$-dimensional balanced complexes form a monoid under coordinate-wise addition. To see this, given $(k-1)$-dimensional balanced complexes $\Delta$ and $\Sigma$ form the *color-preserving product complex* $\Delta \oplus \Sigma$ as follows. Let

$$V := \big\{(d, s) : d \text{ is a vertex of } \Delta,\ s \text{ is a vertex of } \Sigma, \text{ and } c_\Delta(d) = c_\Sigma(e)\big\},$$

define $\delta \oplus \sigma := \{(d, s) \in V : d \in \delta \text{ and } s \in \sigma\}$, and finally let

$$\Delta \oplus \Sigma := \big\{\delta \oplus \sigma : c_\Delta(\delta) = c_\Sigma(\sigma),\ \delta \in \Delta, \text{ and } \sigma \in \Sigma\big\}.$$

One can easily show that $\eta_S(\Delta \oplus \Sigma) = \eta_S(\Delta) + \eta_S(\Sigma)$ for each $S \subseteq [k]$. Can we describe any interesting minimal generating sets? More generally, what can we say about the algebraic structure of this monoid?

**Problem 8.2.** We mention that the cones $H$ and $I$ of flag $\eta$-vectors are polyhedral, but the inequalities we define these cones with, namely the multicover inequalities, form an infinite set. What is a minimal set of inequalities which define the cone? Equivalently, what are its facets?

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## References

[1] F. R. K. Chung, R. L. Graham, P. Frankl and J. B. Shearer: Some intersection theorems for ordered sets and graphs, *J. Combin. Theory Ser. A* **43(1)** (1986), 23–37.

[2] Péter Frankl, Zoltán Füredi and Gil Kalai: Shadows of colored complexes, *Math. Scand.* **63(2)** (1988), 169–178.

[3] Gy. Katona: A theorem of finite sets, in: *Theory of graphs (Proc. Colloq., Tihany, 1966)*, Academic Press, Budapest, New York, 1968, pp. 187–207.

[4] Joseph B. Kruskal: The number of simplices in a complex, in: *Mathematical optimization techniques*, Univ. of California Press, Berkeley, CA, 1963, pp. 251–278.

[5] Alexander Schrijver: *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons Ltd., Chichester, 1986, A Wiley-Interscience Publication.

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